数理方程笔记 Chapter 4

Chaper 4 波动方程

4.1 波动算子的基本解

所谓波动方程即

$$\frac{\partial^2 u}{\partial t^2} - a^2\Delta u = f(x,t)$$

其中 $f(x,t)$ 为外力，当 $n=3$ 时，设下面基本解为 $E(x,t)$:

$$\frac{\partial^2}{\partial t^2} - \Delta$$

即

$$\frac{\partial^2 E(x,t)}{\partial t^2} - a^2\Delta E(x,t) = \delta(x,t)$$

还是一样，对 x 进行 Fourier 变换，得到

$$\frac{\partial^2 \widehat{E}(\xi,t)}{\partial t^2} + a^2|\xi|^2\widehat{E}(\xi,t) = \delta(t)$$

这是一个非齐次的二阶线性 ODE，ODE知识告诉我们

$$\widehat{E}(\xi,t) = k_1(\xi,t)\sin(a|\xi|t) + k_2(\xi,t)\cos(a|\xi|t)$$

对 t 求导有

$$\frac{\partial \widehat{E}}{\partial t} = k_1'\sin(a|\xi|t) + k_2\cos(a|\xi|t)a|\xi| + k_2'\cos(a|\xi|t) - k_2\sin(a|\xi|t)a|\xi|$$

我们人为令

$$k_1'\sin(a|\xi|t) + k_2'\cos(a|\xi|t) = 0$$

于是我们有

$$\frac{\partial^2 \widehat{E}}{\partial t^2} = k_1'\cos(a|\xi|t)a|\xi| - k_1\sin(a|\xi|t)a^2|\xi|^2 - k_2'\sin(a|\xi|t)a|\xi| - k_2\cos(a|\xi|t)a^2|\xi|^2$$

于是得到

$$\frac{\partial^2 \widehat{E}}{\partial t^2} + a^2|\xi|^2\widehat{E} = k_1'\cos(a|\xi|t)a|\xi| - k_2'\sin(a|\xi|t)a|\xi|$$

于是希望

$$k_1'\cos(a|\xi|t)a|\xi| - k_2'\sin(a|\xi|t)a|\xi| = \delta(t)$$

所以希望解的方程为:

$$\begin{cases} k_1'\sin(a|\xi|t) + k_2'\cos(a|\xi|t) = 0 \\ k_1'\cos(a|\xi|t)a|\xi| - k_2'\sin(a|\xi|t)a|\xi| = \delta(t) \end{cases}$$

第一个式子 $\times\cos(a|\xi|t)a|\xi|$ 再减去第二个式子 $\times\sin(a|\xi|t)$ 得到

$$k_2'(\cos^2(a|\xi|t) + \sin^2(a|\xi|t))a|\xi| = -\delta(t)\sin(a|\xi|t)$$

于是得到

$$k_2'a|\xi| = 0 \implies k_2' = 0$$

于是对于变量 t 而言，k_2 是一个常数. 我们取 $k_2 = 0$，得到

$$\begin{cases} k_1'\sin(a|\xi|t) = 0 \\ k_1'\cos(a|\xi|t)a|\xi| = \delta(t) \end{cases}$$

同理可得到

$$k_1'a|\xi| = \delta(t)\cos(a|\xi|t) = \delta(t)$$

即

$$k_1' = \frac{\delta(t)}{a|\xi|}$$

于是

$$k_1(\xi,t) = \begin{cases} \dfrac{H(t)}{a|\xi|}, & t>0 \\ 0, & t=0 \\ \dfrac{-H(-t)}{a|\xi|}, & t<0 \end{cases}$$

于是我们得到

$$\widehat{E}(\xi,t) = \begin{cases} \widehat{E}_+ := \dfrac{1}{a|\xi|}H(t)\sin(a|\xi|t), & t>0 \\ \widehat{E}_- := \dfrac{(-1)}{a|\xi|}H(-t)\sin(a|\xi|t), & t<0 \end{cases}$$

为了求出 $E(x,t)$，我们有两种方法.

法1： 设曲面

$$S = \{x \mid p(x) = 0, \mathrm{d}p(x) \neq 0, p \in C^\infty\}$$

我们定义广义函数

$$\delta(p(x)) \in \mathscr{D}'(\mathbb{R})$$

为

$$\langle\delta(p(x)),\varphi\rangle:=\int_{p(x)=0}\varphi(x)\mathrm{d}s,\quad\forall\varphi\in C_0^\infty(\mathbb{R}^n)$$

我们注意到

$$\operatorname{sing\,supp}\delta(p(x))=\operatorname{supp}\delta(p(x))=\{x\mid p(x)=0\}$$

额外的，如果是紧集，则 $\delta(p(x))\in\mathscr{E}'\subseteq\mathscr{S}'$. 于是有

$$\delta\widehat{(p(x)})=\langle\delta(p(x)),e^{-ix\xi}\rangle=\int_{p(x)=0}e^{-ix\xi}\mathrm{d}s_x$$

当 $n=3$ 的时候，设

$$p(x)=r-|x|$$

则有

$$S=\{x\colon|x|=r\}$$

有

$$x\cdot\xi=|x||\xi|\cos\theta$$

令 $x_{\geq 0}$ 轴经过 ξ 点，有

$$\begin{aligned}\hat{\delta}(r-|x|)(\xi)&=\int_{|x|=r}e^{-i|x||\xi|\cos\theta}\mathrm{d}s_x\\&=\int_0^{2\pi}\int_0^{\pi}e^{-i|x||\xi|\cos\theta}r^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\\&=-2\pi r^2\int_{-1}^{1}e^{-r|\xi|y}\mathrm{d}y\\&=\frac{4\pi r}{|\xi|}\sin(r|\xi|)\end{aligned}$$

当 $t>0$ 时，令 $r=at(a>0)$，我们有

$$\begin{aligned}\widehat{E}_+(\xi,t)&=\frac{1}{a|\xi|}H(t)\sin(a|\xi|t)\\&=\frac{4\pi at}{|\xi|}\left(\frac{1}{a}H(t)\right)\sin(a|\xi|t)\frac{1}{4\pi at}\\&=\frac{1}{4\pi a^2t}H(t)\left(\frac{4\pi at}{|\xi|}\sin(a|\xi|t)\right)\\&=\frac{1}{4\pi a^2t}H(t)\hat{\delta}(r-|x|)(\xi)\end{aligned}$$

于是

$$E_+(x,t)=\frac{1}{4\pi a^2t}H(t)\delta(at-x)$$

当 $t<0$ 时，令 $r=-at>0(a>0)$，同理有

$$\widehat{E}_-(\xi,t)=\frac{-1}{4\pi a^2t}H(-t)\hat{\delta}(at+|x|)(\xi)$$

于是

$$E_-(x,t)=\frac{1}{4\pi a^2 t}(-H(-t))\delta(at+|x|)$$

注：$\begin{cases}\text{supp } E_+(x.t)=\text{sing supp}E_+(x.t)=\{x\big||x|=at\}\\ \text{supp } E_-(x.t)=\text{sing supp}E_-(x.t)=\{x\big||x|=-at\}\end{cases}$

法2： 利用 Fourier 逆变换直接计算，以 $n=3$ 为例：

球坐标变换×ρ²sinθ，|ξ|=ρ，积化和差，极限情况写成δ

$$E_+(x,t)=F^{-1}\widehat{E_+}(\xi,t)=(2\pi)^{-3}H(t)\int_{\mathbb{R}^3}\frac{\sin(a|\xi|t)}{a|\xi|}e^{ix\xi}\mathrm{d}\xi$$

其中，$x\cdot\xi=|x|\cdot|\xi|\cdot\cos\theta$，令$\xi$轴过$x$点，记$|\xi|=\rho$

$$\begin{aligned}
\Rightarrow E_+(x,t)&=(2\pi)^{-3}H(t)\int_0^{+\infty}\frac{\sin(a\rho t)}{a\rho}\rho^2\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}e^{i|x|\rho\cos\theta}\sin\theta\mathrm{d}\theta\\
&\xlongequal{\cos\theta=y}(2\pi)^{-3}H(t)\frac{2\pi}{a}\int_0^{\infty}\sin(a\rho t)\rho\mathrm{d}\rho\int_1^{-1}\cos(|x|\rho y)+i\sin(|x|\rho y)(-1)\mathrm{d}y\\
&=\frac{H(t)}{a|x|}(2\pi)^{-2}\int_0^{+\infty}\sin(a\rho t)2\cdot\sin(|x|\rho)\mathrm{d}\rho\\
&=\frac{H(t)}{4\pi^2 a|x|}\int_0^{+\infty}\cos(|x|-at)\rho-\cos(|x|+at)\rho\mathrm{d}\rho\\
&=\frac{H(t)}{4\pi^2 a|x|}\lim_{A\to+\infty}\frac{\sin[A(|x|-at)]}{|x|-at}-\frac{\sin[A(|x|+at)]}{|x|+at}\\
&\xlongequal{\lim\limits_{n\to\infty}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin nx}{x}\mathrm{d}x=\delta(x)}\frac{H(t)}{4\pi a|x|}(\delta(|x|-at)-\delta(|x|+at))\\
&\xlongequal{t>0\text{时}|x|+at>0\text{所以}\delta(|x|+at)=0}\frac{H(t)}{4\pi a|x|}\delta(|x|-at),t>0
\end{aligned}$$

由 $\text{supp }\delta=\{|x|=at\}\Rightarrow E_+(x,t)=H(t)\dfrac{1}{4\pi a^2 t}\delta(at-|x|),t>0$

$E_-(x,t)=-\dfrac{H(-t)}{4\pi a|x|}\delta(|x|+at),t<0$的情况完全类似

4.2 一维波动方程

4.2.1 初值问题

我们考虑方程

$$(P):\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-a^2\dfrac{\partial^2 u}{\partial x^2}=f(x,t),\quad t>0,x\in\mathbb{R}\\ u|_{t=0}=\varphi(x)\\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x)\end{cases}$$

利用叠加原理，我们只需要分别考虑两个方程：

$$(P_1):\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-a^2\dfrac{\partial^2 u}{\partial x^2}=0, \quad t>0, x\in\mathbb{R}\\ u|_{t=0}=\varphi(x)\\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x)\end{cases}, \quad (P_2):\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-a^2\dfrac{\partial^2 u}{\partial x^2}=f(x,t), \quad t>0, x\in\mathbb{R}\\ u|_{t=0}=0\\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=0\end{cases}$$

现在我们来求解 (P_1)：我们作换元

$$\xi=x-at, \quad \eta=x+at$$

于是我们有

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}=\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}$$

于是

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial \xi^2}+2\frac{\partial^2 u}{\partial \xi\partial \eta}+\frac{\partial^2 u}{\partial \eta^2}$$

而

$$\frac{\partial u}{\partial t}=a\left(\frac{\partial u}{\partial \eta}-\frac{\partial u}{\partial \xi}\right)$$

从而

$$\frac{\partial^2 u}{\partial t^2}=a^2\left(\frac{\partial^2 u}{\partial \xi^2}-2\frac{\partial^2 u}{\partial \xi\partial \eta}+\frac{\partial^2 u}{\partial \eta^2}\right)$$

所以我们得到

$$0=\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=-4a^2\frac{\partial^2 u}{\partial \xi\partial \eta}$$

所以

$$\frac{\partial^2 u}{\partial \xi\partial \eta}=0$$

于是

$$\frac{\partial u}{\partial \xi}=\widetilde{F}(\xi)$$

于是

$$u(\xi,\eta)=F(\xi)+G(\eta)$$

从而

$$u(x,t)=F(x-at)+G(x+at)$$

结合我们的初值条件：

$$\begin{cases} u|_{t=0} = \varphi(x) \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \psi(x) \end{cases}$$

于是我们有

$$\begin{cases} F(x) + G(x) = \varphi(x) \\ a\left(-F'(x) + G'(x)\right) = \psi(x) \end{cases}$$

于是有

$$\begin{cases} F(x) + G(x) = \varphi(x) \\ a\left(-F(x) + G(x)\right) + C = \displaystyle\int_{x_0}^{x} \psi(y)\mathrm{d}y \end{cases}$$

解方程得到

$$\begin{cases} F(x) = \dfrac{1}{2}\varphi(x) - \dfrac{1}{2a}\displaystyle\int_{x_0}^{x} \psi(y)\mathrm{d}y + \dfrac{c}{2a} \\ G(x) = \dfrac{1}{2}\varphi(x) + \dfrac{1}{2a}\displaystyle\int_{x_0}^{x} \psi(y)\mathrm{d}y - \dfrac{c}{2a} \end{cases}$$

于是

$$u(x,t) = \frac{1}{2}(\varphi(x-at) + \varphi(x+at)) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(y)\mathrm{d}y$$

现在来求解 (P_2)，即

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2\dfrac{\partial^2 u}{\partial x^2} = f(x,t), \quad t > 0 \\ u|_{t=0} = 0 \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0 \end{cases}$$

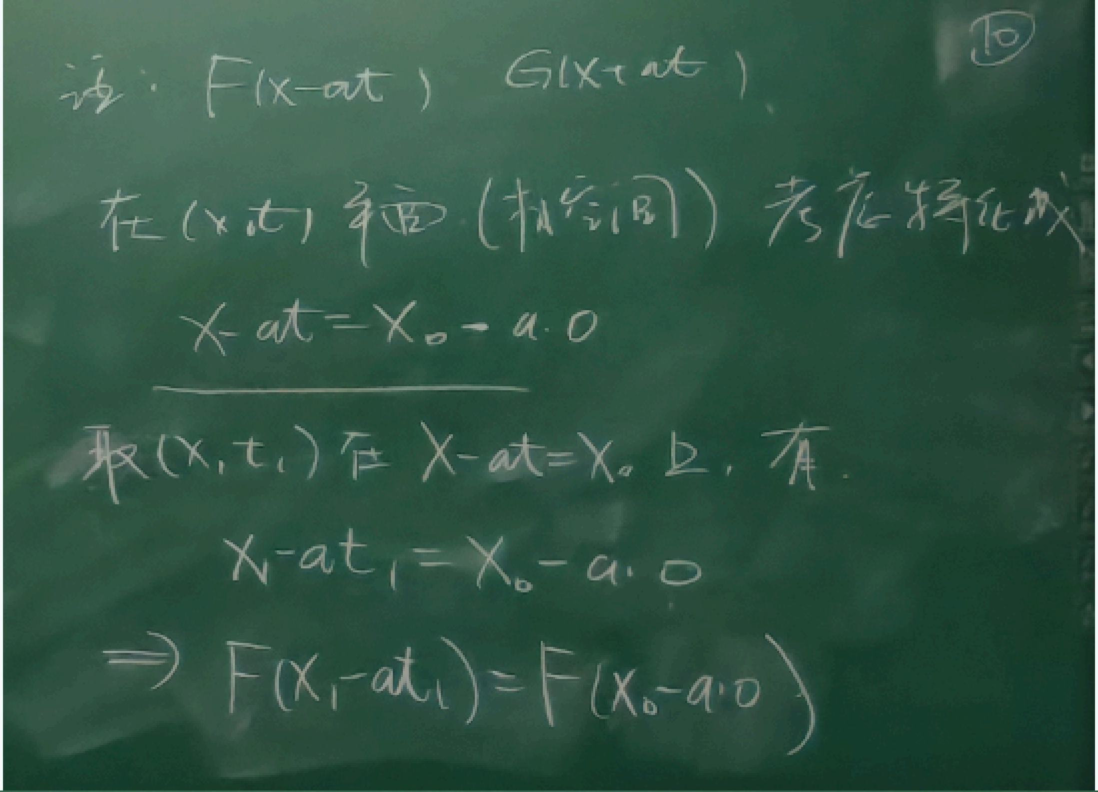
我们使用 Duhamel 原理将其变成一个齐次化的问题

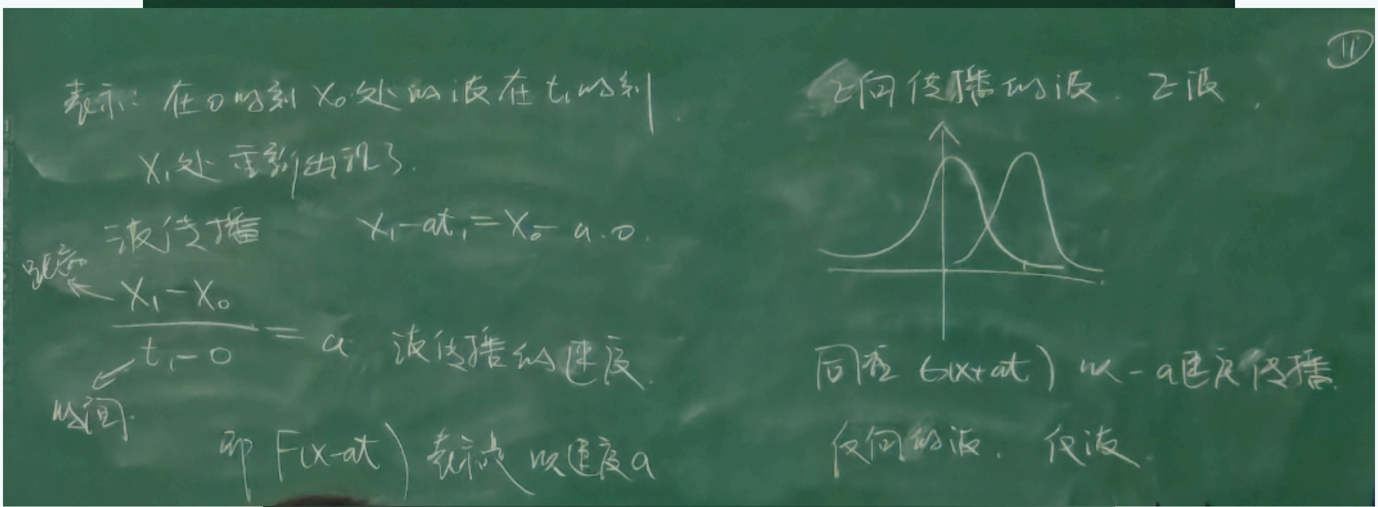
【*Duhamel*原理】令$W(x,t,\tau)$满足$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - a^2\Delta\right)W = 0, \quad t > \tau \\ W(x,t,\tau)|_{t=\tau} = 0 \\ \left.\dfrac{\partial W}{\partial t}\right|_{t=\tau} = f(x,\tau) \end{cases}$, 令$u(x,t) = \displaystyle\int_0^t W(x,t;\tau)\,d\tau$

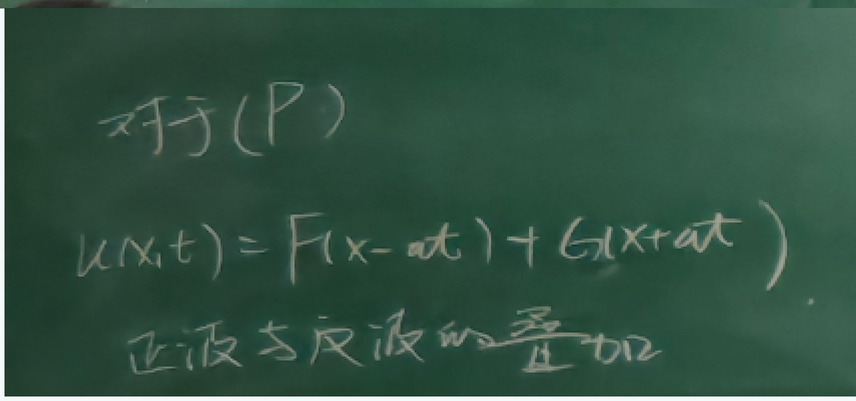
得$W(x,t,\tau) = \dfrac{1}{2a}\displaystyle\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(y,\tau)\mathrm{d}y$, 则$u_2(x,t) = \dfrac{1}{2a}\displaystyle\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(y,\tau)\mathrm{d}y\mathrm{d}\tau$

$\Rightarrow (P_0)$的解为$u(x) = \dfrac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} \psi(y)\mathrm{d}y + \dfrac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(y,\tau)\mathrm{d}y\mathrm{d}\tau$

⚠️ **注意:**







4.2.2 弦振动方程的初边值问题

考虑方程

$$
(P_d)\colon \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t), \quad t>0 \\ u|_{t=0} = \varphi(x), \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \psi(x) \\ u(0,t)=u(l,t)=0 \end{cases}
$$

还是使用叠加原理:

$$(P_{d_1}):\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-a^2\dfrac{\partial^2 u}{\partial x^2}=0, \quad t>0\\ u|_{t=0}=\varphi(x),\\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x)\\ u(0,t)=u(l,t)=0\end{cases}$$

与

$$(P_{d_2}):\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-a^2\dfrac{\partial^2 u}{\partial x^2}=f(x,t), \quad t>0\\ u|_{t=0}=0,\\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=0\\ u(0,t)=u(l,t)=0\end{cases}$$

使用分离变量法求解 (P_{d_1})，假设

$$u(x,t)=X(x)T(t)$$

带入方程就会得到

$$\frac{X''(x)}{X(x)}=\frac{T''(t)}{a^2T(t)}=-\mu$$

变成 S-L 问题:

$$\begin{cases}X''(x)+\mu X(x)=0\\ X(0)=X(l)=0\end{cases}$$

对 μ 分情况讨论：当 $\mu<0$ 时，有

$$X(x)=C_1e^{\sqrt{-\mu}x}+C_2e^{-\sqrt{-\mu}x}$$

代入初值得到

$$C_1=C_2=0$$

平凡解，舍去. 当 $\mu=0$ 时，有

$$X(x)=C_1+C_2x$$

代入得到

$$C_1=C_2=0$$

平凡解，舍去. 当 $\mu>0$ 时，有

$$X(x)=C_1\cos(\sqrt{\mu}x)+C_2\sin(\sqrt{\mu}x)$$

代入 $X(0)=0$ 得到

$$C_1=0$$

代入 $X(l)=0$ 得到

$$C_2\sin(\sqrt{\mu}l)=0$$

故所有非平凡解为

$$X_k(x)=C_k\sin\frac{k\pi}{l}x,\quad \mu_k=\left(\frac{k\pi}{l}\right)^2$$

将 μ_k 代入得到

$$T_k''(t)+a^2\mu_kT_k(t)=0$$

即

$$T_k(t)=\overline{A_k}\cos\left(\frac{k\pi a}{l}t\right)+\overline{B_k}\sin\left(\frac{k\pi a}{l}t\right)$$

于是

$$u_k(x,t)=X_k(x)+T_k(t)$$

从而

$$u(x,t)=\sum_{k=1}^{\infty}u_k(x,t)=\sum_{k=1}^{\infty}\left(A_k\cos\left(\frac{k\pi a}{l}t\right)+B_k\sin\left(\frac{k\pi a}{l}t\right)\right)\sin\left(\frac{k\pi}{l}x\right)$$

利用初值 $u|_{t=0}=\varphi(x)$ 与 $\left.\frac{\partial u}{\partial t}\right|_{t=0}=\psi(x)$ 来确定 A_k,B_k，有

$$\begin{cases}\varphi(x)=\sum\limits_{k=1}^{\infty}A_k\sin\left(\frac{k\pi}{l}x\right)\\ \psi(x)=\sum\limits_{k=1}^{\infty}B_k\frac{k\pi a}{l}\sin\left(\frac{k\pi}{l}x\right)\end{cases}$$

所以最后我们得到:

$$A_k=\frac{2}{l}\int_0^l\varphi(x)\sin\left(\frac{k\pi x}{l}\right)\mathrm{d}x,\quad B_k=\frac{2}{k\pi a}\int_0^l\psi(x)\sin\left(\frac{k\pi x}{l}\right)\mathrm{d}x$$

4.3 $n=2,3$ Cauchy 问题