1 不定积分的引入

1.

$$\int \frac{1}{x}\mathrm{d}x = \ln|x| + C,\ x \neq 0$$

准确说，应该写成：

$$\int \frac{1}{x}\mathrm{d}x = \begin{cases} \ln(-x) + C_1,\ x < 0 \\ \ln x + C_2,\ x > 0 \end{cases}$$

但我们常常略去等号成立条件，只写成最上面的形式.

2.

$$\int \sinh x\mathrm{d}x = \cosh x + C$$

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$$\int \frac{1}{\cosh^2 x}\mathrm{d}x = \tanh x + C$$

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C = -\arccos x + C$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C = -\mathrm{arccot} x + C$$

3.

$$\int \frac{x^2}{1+x^2}\mathrm{d}x = \int 1\mathrm{d}x - \int \frac{\mathrm{d}x}{1+x^2} = x - \arctan x + C$$

4.需要指出：

$$\frac{\mathrm{d}}{\mathrm{d}x}\int f(x)\mathrm{d}x = f(x)$$

$$\mathrm{d}\int f(x)\mathrm{d}x = f(x)\mathrm{d}x$$

$$\int \mathrm{d}f(x) = \int f'(x)\mathrm{d}x = f(x) + C$$

2 换元积分:第一积分换元法(凑微分)，第二积分换元法

5.需要知道：

$$\mathrm{d}x = \frac{1}{a}\mathrm{d}(ax+b), a \neq 0$$

$$x^{\alpha}\mathrm{d}x = \frac{1}{\alpha+1}\mathrm{d}(x^{\alpha+1}), \alpha \neq -1$$

$$\frac{1}{x}\mathrm{d}x = \mathrm{d}(\ln|x|)$$

$$a^x \mathrm{d}x = \frac{1}{\ln a}\mathrm{d}(a^x), a > 0, a \neq 1$$

$$e^x \mathrm{d}x = \mathrm{d}(e^x), e^{-x} = -\mathrm{d}(e^{-x})$$

$$\sec^2 x \mathrm{d}x = \mathrm{d}(\tan x)$$

$$\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \mathrm{d}(\arcsin x)$$

$$\frac{\mathrm{d}x}{1+x^2} = \mathrm{d}(\arctan x)$$

$$\sinh x \mathrm{d}x = \mathrm{d}(\cosh x), \cosh \mathrm{d}x = \mathrm{d}(\sinh x)$$

$$\frac{1}{\cosh^2 x}\mathrm{d}x = \mathrm{d}(\tanh x)$$

$$\frac{\mathrm{d}x}{x\ln x} = \mathrm{d}(\ln\ln x)$$

$$\left(1-\frac{1}{x^2}\right)\mathrm{d}x = \mathrm{d}\left(x+\frac{1}{x}\right)$$

$$\left(1+\frac{1}{x^2}\right)\mathrm{d}x = \mathrm{d}\left(x-\frac{1}{x}\right)$$

$$(x+1)e^x \mathrm{d}x = \mathrm{d}(xe^x), (\ln x+1)\mathrm{d}x = \mathrm{d}(x\ln x)$$

6.

$$\int \frac{x}{1+x^4}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(x^2)}{1+(x^2)^2} = \frac{1}{2}\arctan x^2 + C$$

$$\int \tan x \mathrm{d}x = \int \frac{\sin x \mathrm{d}x}{\cos x} = \int \frac{\mathrm{d}(\cos x)}{\cos x} = -\ln|\cos x| + C$$

$$\int \frac{x^2-x^4}{(x^2+1)^4}\mathrm{d}x = \int \frac{\frac{1}{x^2}-1}{(x+\frac{1}{x})^4}\mathrm{d}x = -\int \frac{\mathrm{d}(x+\frac{1}{x})}{(x+\frac{1}{x})^4} = \frac{1}{3(x+\frac{1}{x})^3} + C = \frac{x^3}{3(x^2+1)^3} + C$$

注：这里的定义域实际上发生了改变，但实际上并不影响，事实上，我们有：

$$Proposition:\ f, F \in C(-\infty, +\infty), F'(x) = f(x)\ \ if\ \ x \neq 0.\ then\ F'(0) = f(0)$$

$$Proof:\ F'_+(0) = \lim_{x\to 0^+}\frac{F(x)-F(0)}{x-0} = \lim_{x\to 0^+}F'(\theta x) = \lim_{x\to 0^+}f(\theta x) = f(0).$$

$$Similarly,\ F'_-(0) = f(0).$$

$$Thus, F'(0) = f(0).$$

7.

$$\int \frac{\mathrm{d}x}{a^2-x^2}(a>0) = \frac{1}{2a}\int\left(\frac{1}{a-x}+\frac{1}{a+x}\right)\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

$$\int \sec x \mathrm{d}x = \int \frac{\cos x}{\cos^2 x}\mathrm{d}x = \int \frac{\mathrm{d}(\sin x)}{1-\sin^2 x} = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C = \ln|\sec x + \tan x| + C$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}(a>0)=\int \frac{\mathrm{d}\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}}=\arcsin\frac{x}{a}+C$$

$$\int \frac{\mathrm{d}x}{1+e^x}=\int \frac{e^{-x}\mathrm{d}x}{1+e^{-x}}=-\int \frac{\mathrm{d}(1+e^{-x})}{1+e^{-x}}=-\ln|1+e^{-x}|+C$$

8.需要知道的：

$$\sqrt{1-\sin^2 x}=\cos x,\ x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\sqrt{1+\tan^2 x}=\sec x,\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\sqrt{\sec^2 x-1}=\begin{cases}\tan x,\ x\in[0,\frac{\pi}{2}),\\ -\tan x,\ x\in(\frac{\pi}{2},\pi]\end{cases}$$

$$\sqrt{\cosh^2 t-1}=\sinh t, t\geq 0$$

$$\sqrt{1+\sinh^2 t}=\cosh t, t\in\mathbb{R}$$

9.

$$\int \sqrt{a^2-x^2}\mathrm{d}x(a>0)\overset{x=a\sin t}{=}a^2\int \cos^2 t\mathrm{d}t=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}(a>0,|x|>a)\overset{x=a\sec t}{=}\int \frac{a\sec t\tan t}{\pm a\tan t}\mathrm{d}t=\pm\int \sec t\mathrm{d}t=\pm\ln|\sec t+\tan t|+C$$

$$=\begin{cases}\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C=\ln|x+\sqrt{x^2-a^2}|+C, x>a\\ -\ln\left|\frac{x}{a}-\frac{\sqrt{x^2-a^2}}{a}\right|+C=-\ln|x-\sqrt{x^2-a^2}|+C, x<-a\end{cases}=\ln|x+\sqrt{x^2-a^2}|+C$$

10.

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}(x>1)=-\int \frac{-\frac{1}{x^2}\mathrm{d}x}{\sqrt{1-\frac{1}{x^2}}}=-\int \frac{\mathrm{d}(\frac{1}{x})}{\sqrt{1-\frac{1}{x^2}}}=-\arcsin\frac{1}{x}+C$$

11.something special:

$$\int \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}}(a<x<b), use\ \ x=a\cos^2 t+b\sin^2 t, 0<t<\frac{\pi}{2}$$

$$\int \frac{\mathrm{d}x}{(x+a)^m(x+b)^n}(m,n\in\mathbb{N}^*), use\ \ t=\frac{x+a}{x+b}$$

3 分部积分

12.

$$\int f(x)g(x)\mathrm{d}x=F(x)g(x)-\int F(x)g'(x)\mathrm{d}x$$

$$\int g(x)\mathrm{d}F(x)=F(x)g(x)-\int F(x)\mathrm{d}g(x)$$

13.

$$\int \ln x\mathrm{d}x=\int \ln x(1)\mathrm{d}x=x\ln x-\int x\frac{1}{x}\mathrm{d}x=x\ln x-x+C$$

可以利用递推求某些不定积分：

$$\int \ln^m x\mathrm{d}x=x\ln^m x-m\int \ln^{m-1}x\mathrm{d}x$$

$$\int x^n\ln^m x\mathrm{d}x=\frac{x^{n+1}\ln^m x}{n+1}-\frac{m}{n+1}\int x^n\ln^{m-1}x\mathrm{d}x$$

14.

$$\int x\cos x\mathrm{d}x=\int x\mathrm{d}\sin x=x\sin x-\int \sin x\mathrm{d}x=x\sin x+\cos x+C$$

进一步考虑一下几个积分递推，可以计算下面几个积分的通项：

$$\int x^{2n}\sin x\mathrm{d}x=-x^{2n}\cos x+2n\int x^{2n-1}\cos x\mathrm{d}x$$

$$\int x^{2n-1}\cos x\mathrm{d}x=x^{2n-1}\sin x-(2n-1)\int x^{2n-2}\sin x\mathrm{d}x$$

$$\int x^{2n-2}\sin x\mathrm{d}x=-x^{2n-2}\cos x+(2n-2)\int x^{2n-3}\cos x\mathrm{d}x$$

$$\int x^{2n-3}\cos x\mathrm{d}x=x^{2n-3}\sin x-(2n-3)\int x^{2n-4}\sin x\mathrm{d}x$$

同理可以有：

$$\int x^n e^{ax}\mathrm{d}x=\frac{x^n e^{ax}}{a}-\frac{n}{a}\int x^{n-1}e^{ax}\mathrm{d}x$$

15.

$$\int e^{ax}\cos bx\mathrm{d}x=\frac{1}{a}\int \cos bx\mathrm{d}e^{ax}=\frac{1}{a}e^{ax}\cos bx+\frac{b}{a}\int e^{ax}\sin bx\mathrm{d}x=\frac{1}{a}e^{ax}\cos bx+\frac{b}{a^2}\int \sin bx\mathrm{d}e^{ax}$$

$$=\frac{1}{a}e^{ax}\cos bx+\frac{b}{a^2}e^{ax}\sin bx-\frac{b^2}{a^2}\int e^{ax}\cos bx\mathrm{d}x$$

移项得：

$$\int e^{ax}\cos bx\mathrm{d}x=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C$$

注[1]：这里两次分部积分要选择同一函数类型，否则不能达到上面那般“解方程”的效果.

注[2]：上面是选择 e^{ax} 作为分部积分的起点，当然也可以尝试对 $\cos bx$ 进行两次分部积分.

16.

$$\int e^{\sin x}\cdot\frac{x\cos^3 x-\sin x}{\cos^2 x}\mathrm{d}x=\int e^{\sin x}\cos x\mathrm{d}x-\int e^{\sin x}\cdot\frac{\sin x}{\cos^2 x}\mathrm{d}x$$

$$=\int x\mathrm{d}(e^{\sin x})-\int e^{\sin x}\mathrm{d}\left(\frac{1}{\cos x}\right)$$

$$= xe^{\sin x} - \int e^{\sin x}\mathrm{d}x - e^{\sin x}\cdot\frac{1}{\cos x} + \int e^{\sin x}\mathrm{d}x$$
$$= e^{\sin x}(x - \sec x) + C$$

注：这里是分别对两个积分运用分部积分公式，得到了两个相同的结构相互抵消，解决了某些积分无法计算的问题.

17.

$$\int \sqrt{x^2-a^2}\mathrm{d}x(a>0) = x\sqrt{x^2-a^2} - \int x\cdot\frac{x}{\sqrt{x^2-a^2}}\mathrm{d}x = x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}$$
$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\mathrm{d}x - a^2\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$$

移项得：

$$\int \sqrt{x^2-a^2}\mathrm{d}x = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C$$

18.更多关于递推的问题：

(1)

$$I_n = \int\frac{\mathrm{d}x}{(x^2+a^2)^n}, n\in\mathbb{N}^*$$
$$I_n = \frac{1}{a^2}\int\frac{x^2+a^2-x^2}{(x^2+a^2)^n}\mathrm{d}x = \frac{1}{a^2}I_{n-1} - \frac{1}{a^2}\int\frac{x^2}{(x^2+a^2)^n}\mathrm{d}x$$
$$= \frac{1}{a^2}I_{n-1} + \frac{1}{2(n-1)a^2}\int x\left[\frac{1}{(x^2+a^2)^{n-1}}\right]'\mathrm{d}x$$
$$= \frac{1}{a^2}I_{n-1} + \frac{1}{2(n-1)a^2}\left[\frac{x}{(x^2+a^2)^{n-1}} - \int\frac{\mathrm{d}x}{(x^2+a^2)^{n-1}}\right]$$
$$= \frac{2n-3}{2(n-1)a^2}I_{n-1} + \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}}$$

(2)

$$I_n = \int \sin^n x\mathrm{d}x(n\in\mathbb{N}^*) = -\int \sin^{n-1}x\mathrm{d}(\cos x)$$
$$= -\sin^{n-1}x\cos x + (n-1)\int\sin^{n-2}x(1-\sin^2 x)\mathrm{d}x$$
$$= -\sin^{n-1}x\cos x + (n-1)(I_{n-2} - I_n)$$

可得：

$$I_n = \frac{n-1}{n}I_{n-2} - \frac{1}{n}\sin^{n-1}x\cos x$$

(3)

$$I_n = \int\tan^n x\mathrm{d}x = \int\tan^{n-2}x\frac{\sin^2 x}{\cos^2 x}\mathrm{d}x = \int\tan^{n-2}x\frac{1-\cos^2 x}{\cos^2 x}\mathrm{d}x$$
$$= \int\tan^{n-2}x\mathrm{d}(\tan x) - \int\tan^{n-2}x\mathrm{d}x = \frac{1}{n-1}\tan^{n-1}x - I_{n-2}$$

19.补充积分表：

$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C,\ \ a \neq 0$$

$$\int \frac{\mathrm{d}x}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C,\ \ a \neq 0$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C,\ \ a > 0$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2\pm a^2}} = \ln|x+\sqrt{x^2\pm a^2}| + C$$

$$\int \sqrt{a^2-x^2}\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C,\ \ a > 0$$

$$\int \sqrt{x^2\pm a^2}\mathrm{d}x = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\ln|x+\sqrt{x^2\pm a^2}| + C$$

$$\int \tan x\mathrm{d}x = -\ln|\cos x| + C$$

$$\int \cot x\mathrm{d}x = \ln|\sin x| + C$$

$$\int \sec x\mathrm{d}x = \ln|\sec x + \tan x| + C$$

$$\int \csc x\mathrm{d}x = \ln|\csc x - \cot x| + C$$

4 纯享积分表

$$\int \sec x \tan x \mathrm{d}x = \sec x + C, \qquad \int \csc x \cot x \mathrm{d}x = -\csc x + C$$

$$\int \sec^2 x \mathrm{d}x = \tan x + C, \qquad \int \csc^2 x \mathrm{d}x = -\cot x + C$$

$$\int \sinh x \mathrm{d}x = \cosh x + C, \qquad \int \cosh x \mathrm{d}x = \sinh x + C$$

$$\int \frac{1}{\cosh^2 x} \mathrm{d}x = \tanh x + C$$

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C = -\arccos x + C$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C = -\mathrm{arccot} x + C$$

$$\int \frac{x}{1+x^4} \mathrm{d}x = \frac{1}{2} \arctan x^2 + C$$

$$\int \frac{\mathrm{d}x}{1+e^x} = -\ln|1+e^{-x}| + C$$

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}} (x>1) = -\arcsin \frac{1}{x} + C$$

$$\int \ln x \mathrm{d}x = x \ln x - x + C$$

$$\int x \cos x \mathrm{d}x = x \sin x + \cos x + C$$

$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a} \arctan \frac{x}{a} + C, \ a \neq 0$$

$$\int \frac{\mathrm{d}x}{(a^2+x^2)^2} = \frac{1}{2a^3} \left(\arctan \frac{x}{a} + \frac{ax}{x^2+a^2} \right) + C$$

$$\int \frac{\mathrm{d}x}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C, \ a \neq 0$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C, \ a > 0$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \sqrt{a^2-x^2} \mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C, \ a > 0$$

$$\int \sqrt{x^2 \pm a^2} \mathrm{d}x = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \tan x \mathrm{d}x = -\ln|\cos x| + C$$

$$\int \cot x \mathrm{d}x = \ln|\sin x| + C$$

$$\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C$$

$$\int \csc x \mathrm{d}x = \ln|\csc x - \cot x| + C$$