1.2 2024 年 10 月 10 日

1.2.1 陈氏定理证明的第二步

目标：$R_{1,2}(N) \geq 0.67\frac{\mathfrak{S}_N N}{(\log N)^2}$，其中 $\mathfrak{S}_N = \prod_{p|N,p>2}\frac{p-1}{p-2}\prod_{p>2}\frac{p(p-2)}{(p-1)^2}$.

下面介绍 Selberg 筛法，证明见 Nathanson 的加性数论.

Theorem 1.2.1. 在一些条件下，若 $z << \xi^\lambda, \lambda > 0$，则有上界估计：

$$S_N(\mathcal{C},q,z) \leq \frac{\gamma(q)}{q}\cdot X\cdot\Gamma_N(z)\left(F\left(\frac{\log(\xi^2)}{\log z}\right)+\varepsilon\right)+\sum_{n\leq\xi^2,n|P_N(z)}e^{v(n)}\eta(x,qn)$$

有下界估计：

$$S_N(\mathcal{C},q,z) \leq \frac{\gamma(q)}{q}\cdot X\cdot\Gamma_N(z)\left(f\left(\frac{\log(\xi^2)}{\log z}\right)-\varepsilon\right)-\sum_{n\leq\xi^2,n|P_N(z)}e^{v(n)}\eta(x,qn)$$

Remark 1.2.1.

- 其中 $\gamma(x)$ 为乘性函数，使得

$$\eta(x,n) = \left|\sum_{a\in\mathcal{C},n|a}1-\frac{\gamma(n)}{n}X\right|$$

较小，其中 $X \sim |\mathcal{C}|$.

即选择

$$\gamma(p) = \begin{cases}\frac{p}{p-1}, & p\nmid N\\ 0, & p\mid N\end{cases}$$

- $\Gamma_N(z) = \prod_{p<z,p\nmid N}\left(1-\frac{\gamma(p)}{p}\right)$.
- $v(n)$ 为 n 的不同的素因子个数.
- $F(u)$ 与 $f(u)$ 由下面的微分方程定义

$$\begin{cases}F(u)=\frac{2e^{\gamma_0}}{u}, f(u)=0, & 0<u\leq 2\\ (uF(u))'=f(u-1),(uf(u))'=F(u-1), & 2\leq u\end{cases}$$

- γ_0 为欧拉常数.

则我们有

$$\begin{aligned}
\Gamma_N(z) = \prod_{p<z, p\nmid N}\left(1-\frac{\gamma(p)}{p}\right) &\sim \frac{N}{\varphi(N)}\prod_{p\nmid N}\frac{1-\frac{\gamma(p)}{p}}{1-\frac{1}{p}}\frac{e^{-\gamma_0}}{\log z} \\
(\text{在 } \mathcal{A} \text{ 情况下}) &\sim \frac{N}{\varphi(N)}\prod_{p\nmid N}\frac{p(p-2)}{(p-1)^2}\frac{e^{-\gamma_0}}{\log z} \\
&\sim \frac{N}{\varphi(N)}\prod_{p\mid N, p>2}\frac{(p-1)^2}{p(p-2)}\cdot\prod_{p>2}\frac{p(p-2)}{(p-1)^2}\frac{e^{-\gamma_0}}{\log z} \\
&\sim 2\left(\prod_{p\mid N, p>2}\frac{p-1}{p-2}\right)\cdot\prod_{p>2}\frac{p(p-2)}{(p-1)^2}\frac{e^{-\gamma_0}}{\log z} \\
&\sim \frac{2e^{-\gamma_0}}{\log z}\mathfrak{S}_N
\end{aligned}$$

所以我们可以估计[3]

$$\begin{aligned}
S_N(\mathcal{A}, 1, N^{\frac{1}{10}}) &\geq \frac{N}{\log N}\cdot\frac{2e^{-\gamma_0}}{\log(N^{\frac{1}{10}})}\mathfrak{S}_N\left(f\left(\frac{\log(\xi^2)}{\log(N^{\frac{1}{10}})}\right)-\varepsilon\right)+\sum_{n\leq\xi^2, n\mid P_N(z)}3^{v(n)}\eta(x,n) \\
&\geq \frac{20N}{\log N}\frac{e^{-\gamma_0}}{\log N}\mathfrak{S}_N f(5)+E_{\mathcal{A}}
\end{aligned}$$

我们可以解得

$$f(u) = \begin{cases}
0, & 0<u\leq 2 \\
\dfrac{2e^{\gamma_0}\log(u-1)}{u}, & 2<u\leq 4 \\
\dfrac{2e^{\gamma_0}\left(\log(u-1)+\int_3^{u-1}\dfrac{\int_2^{t-1}\frac{\log(s-1)}{s}\mathrm{d}s}{t}\mathrm{d}t\right)}{u}, & 4<u\leq 6
\end{cases}$$

从而我们可以估计出

$$\begin{aligned}
S_N(\mathcal{A}, 1, N^{\frac{1}{10}}) &\geq \frac{20N}{\log N}\frac{e^{-\gamma_0}}{\log N}\mathfrak{S}_N f(5)+E_{\mathcal{A}} \\
&\geq 11.208\frac{N}{(\log N)^2}\mathfrak{S}_N+E_{\mathcal{A}}
\end{aligned}$$

对于第二项的上界估计：

[3] ξ^2 越大越好，但不能太大，从而让余项可控，我们在这里取 $\xi_{\mathcal{A}}^2 = N^{\frac{1}{2}}(\log N)^{-B}$.

我们令 $\xi_{\mathcal{A},p}^2 = N^{\frac{1}{2}} (\log N)^{-B}$，可以求出

$$F(u)=\begin{cases}\dfrac{2e^{\gamma_0}}{u}, & 0<u\le 3\\ \dfrac{2e^{\gamma_0}\left(1+\int_2^{u-1}\dfrac{\log(t-1)}{t}\mathrm{d}t\right)}{u}, & 3<u\le 5\end{cases}$$

则有

$$\begin{aligned}\sum_{N^{\frac{1}{10}}<p<N^{\frac{1}{3}}} S_N(\mathcal{A},p,N^{\frac{1}{10}}) &\le \sum_{N^{\frac{1}{10}}<p<N^{\frac{1}{3}}}\left(\frac{1}{p-1}\frac{N}{\log N}\frac{2e^{-\gamma_0}}{\log N}\mathfrak{S}_N F\left(\frac{\log(\xi_{\mathcal{A},p}^2)}{\frac{1}{10}\log N}\right)+E_{\mathcal{A},p}\right)\\ &\le 17.112\frac{\mathfrak{S}_N N}{(\log N)^2}+E_2\end{aligned}$$

再算：

$$X\sim|B|\sim\frac{N}{\log N}\int_{\frac{1}{10}}^{\frac{1}{3}}\frac{\mathrm{d}\alpha}{\alpha}\int_{\frac{1}{3}}^{\frac{1}{2}-\frac{1}{2}\alpha}\frac{\mathrm{d}\beta}{\beta(1-\alpha-\beta)}$$

有

$$\Gamma_N(z)\sim\prod_{p<N^{\frac{1}{2}(\log N)^{-B}},p\nmid N}\left(1-\frac{\gamma(p)}{p}\right)\sim\frac{4e^{-\gamma_0}\mathfrak{S}_N}{\log N}$$

令 $\xi_B^2 = N^{\frac{1}{2}} (\log N)^{-B} = z$，代入 $F(1) = 2e^{\gamma_0}$，有

$$\begin{aligned}S_N(\mathcal{B},1,N^{\frac{1}{2}}(\log N)^{-B}) &\le 0.491\frac{N}{\log N}\frac{4e^{-\gamma_0}\mathfrak{S}_N}{\log N}2e^{\gamma_0}+E_3\\ &\le 3.928\frac{\mathfrak{S}_N N}{(\log N)^2}+E_3\end{aligned}$$

所以我们得出

$$R_{1,2}(N)\ge\left(11.208-\frac{1}{2}17.112-\frac{1}{2}3.928\right)\frac{\mathfrak{S}_N N}{(\log N)^2}+E_1+E_2+E_3>0.67\frac{\mathfrak{S}_N N}{(\log N)^2}$$

至此，陈氏定理证毕.